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Constructing conditionally integrable evolution systems in $(1+1)$ dimensions: a generalization of invariant modules approach

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Abstract

Given a generalized (Lie–Bäcklund) vector field satisfying certain nondegeneracy assumptions, we explicitly describe all $(1+1)$-dimensional evolution systems that admit this vector field as a generalized conditional symmetry. The connection with the theory of symmetries of systems of ODEs and with the theory of invariant modules is discussed.

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Introduction

Many important partial differential equations arising in applications are not integrable via the inverse scattering transform or explicit linearization and possess a very poor asset of symmetries. However, among these equations we can single out *conditionally* (or *partially*) *integrable* ones [1–6], for which a number of exact solutions can be found. In order to find these solutions, one usually tries to make a substitution which reduces the equation in question to a simpler one, as, for example, in the fabulous direct reduction method of Clarkson and Kruskal [7]. Another strategy is to find the differential equation(s) (*differential constraint(s)* [8]) compatible with the original equation, and then to look for the common solutions of the system made of the original equation and of the differential constraint(s). The concept of *conditional symmetry*, introduced by Bluman and Cole [9] and subsequently generalized by Olver and Rosenau [10, 11], Levi and Winternitz [12], Fushchich *et al* [13], and Fokas and Liu [14], cf also Grundland *et al* [15, 16], provides an appropriate symmetry background for these procedures and their modifications.

To be more specific, consider a $(1+1)$-dimensional evolution system

$$\partial \boldsymbol{u}/\partial t = \boldsymbol{F}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) \qquad n \geqslant 0 \qquad \partial \boldsymbol{F}/\partial \boldsymbol{u}_n \neq 0 \tag{1}$$

for an s-component vector function $\boldsymbol{u} = (u^1, \ldots, u^s)^T$, where $\boldsymbol{u}_l = \partial^l \boldsymbol{u}/\partial x^l$, $l = 0, 1, 2, \ldots,$ $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, and the superscript '$T$' denotes the matrix transposition.

On the solution manifold of (1) we can always express the derivatives $\partial^{l+m}\boldsymbol{u}/\partial t^l \partial x^m$ with $l > 0$ via $\boldsymbol{u}_j$. Hence we lose no generality in considering only differential constraints of the form $\vec{Q}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k) = 0$, where $\vec{Q} = (Q^1, \ldots, Q^b)^T$ (cf, e.g., [1, 14, 17]).

Then the compatibility condition of (1) and $\vec{Q} = 0$ reads (cf, e.g., [14])

$$D_t(\vec{Q})|_{\mathcal{M}} = 0. \tag{2}$$

Here $D_t = \partial/\partial t + \sum_{i=0}^{\infty} D^i(\boldsymbol{F})\partial/\partial \boldsymbol{u}_i$ and $D \equiv D_x = \partial/\partial x + \sum_{i=0}^{\infty} \boldsymbol{u}_{i+1}\partial/\partial \boldsymbol{u}_i$ are total derivatives with respect to t and x, and $\mathcal{M}$ is the solution manifold of the system $\vec{Q} = 0$.

Let us further specify to the case when $b = s$, i.e. when $\vec{Q}$ has the same number of components as the unknown function $\boldsymbol{u}$, and thus $\vec{Q} = 0$ is a determined system. In order to stress that $b = s$, we shall write $\boldsymbol{Q}$ instead of $\vec{Q}$. In this case the generalized vector field $\mathcal{Q} = \boldsymbol{Q}\partial/\partial \boldsymbol{u}$ is nothing but *generalized conditional symmetry* of (1), introduced by Fokas and Liu [14] and independently by Zhdanov [17]. Moreover, as was shown by Zhdanov [18], for scalar evolution equations (1), i.e. when $s = 1$, such symmetry exists if and only if the equation in question admits the reduction to a system of ODEs in time t.

Note that finding generalized conditional symmetries admitted by (1) with fixed $\boldsymbol{F}$ is a highly nontrivial problem. For instance, theorem 1 of [19] states that finding a generalized conditional symmetry of the form $(\boldsymbol{u}_1 - \boldsymbol{\eta}(t, x, \boldsymbol{u}))\partial/\partial \boldsymbol{u}$ for (1) with $s = 1$ is equivalent to finding a one-parametric family of solutions of (1).

In view of the above, the classification of evolution systems (1) possessing a generalized conditional symmetry of the prescribed form is of paramount importance [20]. If $\boldsymbol{Q}$ is independent of t, then (2) is equivalent to the requirement that $\boldsymbol{F}\partial/\partial \boldsymbol{u}$ is a generalized symmetry (depending on an extra parameter t) for the system of ODEs $\boldsymbol{Q} = 0$, and hence the classification in question amounts to finding all generalized symmetries of the system $\boldsymbol{Q} = 0$. For scalar $\boldsymbol{u}$, i.e. for $s = 1$, this problem was solved by Svirshchevskii [21] for the case of linear ODEs $\boldsymbol{Q} = 0$, see also [22, 23]. This case was also considered by Fokas and Liu [24], and by Athorne [25]. The complete description of the set of generalized symmetries for a generic nonlinear ODE was obtained by Athorne in [26]. Part of these results was rediscovered by Doyle [27] and Samokhin [28]. The latter author has also extended these results to the case of systems of ODEs. A deep generalization of the work of Svirshchevskii to the case of multiple space dimensions was made in the seminal paper [29] by Kamran, Milson and Olver. In [21–24, 27, 29] the applications to construction of conditionally integrable evolution systems (1) were presented. Let us mention that in [27] the trivial, i.e. vanishing on the solution manifold $\mathcal{M}$ of $\boldsymbol{Q} = 0$, symmetries of $\boldsymbol{Q} = 0$ were ignored and hence the results of [27] proved to be incomplete.

To the best of our knowledge, no attempts were made to extend the results of [21, 24, 27, 29] to the case of explicitly time-dependent generalized conditional symmetries. This extension is of particular interest, because the class of evolution systems admitting the symmetries of this kind, as well as the asset of exact solutions constructed using the reduction under these symmetries, obviously are considerably larger than those considered in [21–24, 27, 29]. In particular, if we have an exactly solvable system of ODEs $\boldsymbol{Q}(x, \boldsymbol{u}, \ldots, \boldsymbol{u}_k, \alpha_1, \ldots, \alpha_q) = 0$ involving q parameters α_i, we can replace these parameters by arbitrary functions $\alpha_i(t)$ without affecting the exact solvability. Clearly, because of the presence of arbitrary functions $\alpha_i(t)$ the class of evolution systems (1) admitting $\boldsymbol{Q}(x, \boldsymbol{u}, \ldots, \boldsymbol{u}_k, \alpha_1(t), \ldots, \alpha_q(t))\partial/\partial \boldsymbol{u}$ as a generalized conditional symmetry is considerably richer than its counterpart for $\boldsymbol{Q}(x, \boldsymbol{u}, \ldots, \boldsymbol{u}_k, \alpha_1, \ldots, \alpha_q)\partial/\partial \boldsymbol{u}$.

However, if $\boldsymbol{Q}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)\partial/\partial\boldsymbol{u}$ is a generalized conditional symmetry for (1), and $\partial\boldsymbol{Q}/\partial t \neq 0$, then $\boldsymbol{F}\partial/\partial\boldsymbol{u}$ is no longer a generalized symmetry for $\boldsymbol{Q} = 0$, so the results of [21, 24, 27, 29] on the structure of the respective systems (1) do not apply.

In this paper, we solve this problem and generalize the results from [21, 24, 27, 29]. Namely, we provide the complete description of all evolution systems (1) that possess a generalized conditional symmetry of the form $\boldsymbol{Q}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)\partial/\partial\boldsymbol{u}$ under the assumption that the system $\boldsymbol{Q} = 0$ is analytic and totally nondegenerate, see section 1 below and chapter 2 of [1] for the precise definition of these properties, and the general solution of this system can be found.

1. The main result

Suppose that the general solution of $\boldsymbol{Q} = 0$ in implicit form can be written as

$$\boldsymbol{H}(x, t, \boldsymbol{u}, c_1(t), \ldots, c_N(t)) = 0 \tag{3}$$

for some natural number N, which is called the *total order* of the system $\boldsymbol{Q} = 0$. We assume here that $\boldsymbol{H}$ depends on N arbitrary functions $c_i(t)$, $i = 1, \ldots, N$, in an essential way, and that the condition $\det \partial\boldsymbol{H}/\partial\boldsymbol{u} \neq 0$ holds.

Note that provided the system $\boldsymbol{Q} = 0$ is (locally) normal, the number N can be readily computed and is given by (7), see the discussion after lemma 1 for details.

As $\det \partial\boldsymbol{H}/\partial\boldsymbol{u} \neq 0$ by assumption, by the implicit function theorem we can (at least locally) solve (3) for $\boldsymbol{u}$ and obtain the general solution of $\boldsymbol{Q} = 0$ in explicit form:

$$\boldsymbol{u} = \boldsymbol{P}(x, t, c_1(t), \ldots, c_N(t)). \tag{4}$$

Assume that repeatedly differentiating (4) with respect to x and solving the resulting equations and (4) with respect to $c_1, \ldots, c_N$, we can express c_i via $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$:

$$c_i = h_i(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) \qquad i = 1, \ldots, N. \tag{5}$$

Let $\tilde{\boldsymbol{B}}_i(x, t, c_1, \ldots, c_N) = \partial\boldsymbol{P}(x, t, c_1, \ldots, c_N)/\partial c_i$ and $\boldsymbol{B}_i = \tilde{\boldsymbol{B}}_i(x, t, h_1, \ldots, h_N)$, i.e. $\boldsymbol{B}_i$ are obtained from $\tilde{\boldsymbol{B}}_i$ by substitution of h_i instead of c_i. Likewise, let $\tilde{\boldsymbol{R}}(x, t, c_1, \ldots, c_N) = \partial\boldsymbol{P}(x, t, c_1, \ldots, c_N)/\partial t$, and set $\boldsymbol{R}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) \equiv \tilde{\boldsymbol{R}}(x, t, h_1(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r), \ldots, h_N(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r))$. Here it is understood that c_i are *not* differentiated with respect to t while evaluating $\partial\boldsymbol{P}/\partial t$.

Note that by the implicit function theorem $\tilde{\boldsymbol{B}}_i = -(\partial\boldsymbol{H}/\partial\boldsymbol{u})^{-1}\partial\boldsymbol{H}/\partial c_i$ and $\tilde{\boldsymbol{R}} = -(\partial\boldsymbol{H}/\partial\boldsymbol{u})^{-1}\partial\boldsymbol{H}/\partial t$, whence it is immediate that knowing the general solution of $\boldsymbol{Q} = 0$ in implicit form (3) is sufficient for finding $\boldsymbol{B}_i$ and $\boldsymbol{R}$.

Lemma 1. *Let $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ be a generalized conditional symmetry for (1). Then $\tilde{\boldsymbol{F}} = \boldsymbol{F} - \boldsymbol{R}$ is a generalized symmetry for the system $\boldsymbol{Q} = 0$ considered as a system of ODEs.*

Proof. Indeed, substituting $\boldsymbol{F} = \tilde{\boldsymbol{F}} + \boldsymbol{R}$ and $\vec{Q} = \boldsymbol{Q}$ into (2) and taking into account that

$$\left.\left(\partial\boldsymbol{Q}/\partial t + \sum_{i=0}^{k} D^i(\boldsymbol{R})\partial\boldsymbol{Q}/\partial\boldsymbol{u}_i\right)\right|_{\mathcal{M}} \equiv 0$$

we find (cf, e.g., [17]) that (2) reduces to

$$\left.\left(\sum_{i=0}^{k} D^i(\tilde{\boldsymbol{F}})\partial\boldsymbol{Q}/\partial\boldsymbol{u}_i\right)\right|_{\mathcal{M}} = 0$$

which is exactly the determining equation for the generalized symmetries of the system $\boldsymbol{Q} = 0$ considered as a system of ODEs. □

Let W be the set of all points of an open domain V of the space $\mathcal{V}$ of variables $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$ that satisfy the equations $D^j(\boldsymbol{Q}) = 0$, $j = 0, 1, 2, \ldots$, considered as algebraic equations. Suppose that at all points of W the system $\boldsymbol{Q} = 0$ and all its prolongations $D^j(\boldsymbol{Q}) = 0$, $j \in \mathrm{N}$, are locally solvable and have maximal rank, i.e. the system $\boldsymbol{Q}(x, t, \boldsymbol{u}, \ldots, \boldsymbol{u}_k) = 0$, considered as a system of ODEs involving an extra parameter t, is totally nondegenerate on W, cf chapter 2 of [1]. Further assume that $D^j(\boldsymbol{Q})$, $j = 0, 1, 2, \ldots$, are analytic functions of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$ on V. Then by theorem 2.84 from [1] the system $\boldsymbol{Q} = 0$ is normal on W, and hence can (at least locally) be transformed into the form

$$u^\alpha_{n_\alpha} = g^\alpha\left(x, t, u^1, \ldots, u^1_{n_1-1}, \ldots, u^s, \ldots, u^s_{n_s-1}\right) \qquad \alpha = 1, \ldots, s \tag{6}$$

via an appropriate change of variables. In what follows we assume that the number $\tilde{N} = \sum_{\alpha=1}^s n_\alpha$ is the same for the whole W.

Note that under the assumptions made, the solution manifold $\mathcal{M}$ of $\boldsymbol{Q} = 0$ can (again at least locally) be identified with W (cf, e.g., [1]). In particular, $\tilde{N}$ coincides with the total order N of the system $\boldsymbol{Q} = 0$, cf (3), that is,

$$N = \sum_{\alpha=1}^s n_\alpha. \tag{7}$$

The general solution of (6) can be readily written in the form (4). Repeatedly differentiating (4) with respect to x yields

$$u^\alpha_j = \partial^j P^\alpha(x, t, c_1(t), \ldots, c_N(t))/\partial x^j, \qquad j = 1, 2, \ldots, n_\alpha - 1 \quad \alpha = 1, \ldots, s$$

whence we find $c_i = h_i\left(x, t, u^1, \ldots, u^1_{n_1-1}, \ldots, u^s, \ldots, u^s_{n_s-1}\right)$, $i = 1, \ldots, N$.

For the sake of brevity, let us agree that in the subsequent discussion on symmetries of $\boldsymbol{Q} = 0$ and of (6) we shall consider these systems as systems of ODEs involving a parameter t without repeating this each time explicitly.

By theorem 1 and remark 3 of Samokhin [28] any generalized symmetry of (6) that does not vanish on the solution manifold of (6) has the form $\boldsymbol{S}\partial/\partial\boldsymbol{u}$ with

$$\boldsymbol{S} = \sum_{j=1}^N \psi_j(t, h_1, \ldots, h_N)\boldsymbol{B}_j \tag{8}$$

where ψ_j are some smooth functions of their arguments.

Returning from (6) to $\boldsymbol{Q} = 0$, we readily see that (8) describes all nonvanishing on $\mathcal{M}$ generalized symmetries for $\boldsymbol{Q} = 0$ as well, provided the functions h_i are now defined by means of (5), with $\boldsymbol{P}$ being a general solution of $\boldsymbol{Q} = 0$.

However, this does not give *all* symmetries of $\boldsymbol{Q} = 0$. The point is that we should take into account the trivial generalized symmetries of $\boldsymbol{Q} = 0$, which vanish on $\mathcal{M}$. These can be found in the following way. As $\boldsymbol{Q} = 0$ is totally nondegenerate by assumption, any function of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$ depending on a finite number of $\boldsymbol{u}_j$ and vanishing on $\mathcal{M}$ can be written [1] as a linear combination of $D^j(Q^\alpha)$, $\alpha = 1, \ldots, s$, $j = 0, 1, 2, \ldots$, and the coefficients in this linear combination are arbitrary smooth functions of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$.

Thus, the most general expression for the characteristics $\boldsymbol{S}$ of a generalized symmetry $\boldsymbol{S}\partial/\partial\boldsymbol{u}$ of the system $\boldsymbol{Q} = 0$ is given by (cf [29])

$$\boldsymbol{S} = \sum_{i=1}^N \psi_i(t, h_1, \ldots, h_N)\boldsymbol{B}_i + \sum_{p=0}^m \sum_{\alpha=1}^s \boldsymbol{\chi}_{p,\alpha}(x, t, \boldsymbol{u}, \ldots, \boldsymbol{u}_{j_{p,\alpha}}) D^p(Q^\alpha) \tag{9}$$

where m and $j_{p,\alpha}$ are arbitrary nonnegative integers, $\boldsymbol{\chi}_{p,\alpha}$ and ψ_i are arbitrary smooth functions of their arguments, and h_i are defined by (5). Note that in (9) and (10) $\boldsymbol{\chi}_{p,\alpha}$ are s-component vectors.

By lemma 1 $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ is a generalized conditional symmetry for (1) if and only if $(\boldsymbol{F}-\boldsymbol{R})\partial/\partial\boldsymbol{u}$ is a generalized symmetry for the system of ODEs $\boldsymbol{Q}=0$. Together with (9) this yields the following result, which generalizes the classification theorems of Doyle [27] and Svirshchevskii [21–23], and, partially, of Fokas and Liu [24] and of Kamran, Milson and Olver [29].

Proposition 1. *Let V be an open domain in the space $\mathcal{V}$ of variables $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$, and let W be the set of all points in V satisfying the equations $D^j(\boldsymbol{Q})=0$, $j=0,1,2,\ldots$, considered as algebraic equations. Suppose that the system of ODEs $\boldsymbol{Q}(x,t,\boldsymbol{u},\ldots,\boldsymbol{u}_k)=0$ is analytic on V, is totally nondegenerate on W, and has the same total order N on the whole W. Further assume that $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ is a generalized conditional symmetry for the system $\boldsymbol{u}_t=\boldsymbol{F}$ (1).*

Then $\boldsymbol{F}$ on V can be represented in the form

$$\boldsymbol{F}=\boldsymbol{R}+\sum_{i=1}^{N}\zeta_i(t,h_1,\ldots,h_N)\boldsymbol{B}_i+\sum_{p=0}^{m}\sum_{\alpha=1}^{s}\boldsymbol{\chi}_{p,\alpha}(t,x,\boldsymbol{u},\ldots,\boldsymbol{u}_{j_{p,\alpha}})D^p(Q^\alpha) \tag{10}$$

where m and $j_{p,\alpha}$ are some nonnegative integers, ζ_i and $\boldsymbol{\chi}_{p,\alpha}$ are some smooth functions of their arguments, and h_i are defined by (5).

In other words, (10), where ζ_i and $\boldsymbol{\chi}_{p,\alpha}$ now are arbitrary (smooth) functions of their arguments, represents on V the most general $\boldsymbol{F}$ such that the evolution system $\boldsymbol{u}_t=\boldsymbol{F}$ of the form (1) has the conditional symmetry $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ that satisfies the conditions of proposition 1. Of course, if we wish to fix the order n of $\boldsymbol{F}$, then we should take special care to ensure that (10) contains no terms of order higher than n.

Note that on $\mathcal{M}$ we can express $\boldsymbol{u}_i$, $i\geqslant k$, via $x,t,\boldsymbol{u},\ldots,\boldsymbol{u}_{k-1}$, using the equations $D^j(\boldsymbol{Q})=0$ (cf, e.g., [1]). Hence, if we require $\boldsymbol{F}$ to be analytic on V, then we can rewrite the terms vanishing on $\mathcal{M}$, i.e. $\sum_{p=0}^{m}\sum_{\alpha=1}^{s}\boldsymbol{\chi}_{p,\alpha}(t,x,\boldsymbol{u},\ldots,\boldsymbol{u}_{j_{p,\alpha}})D^p(Q^\alpha)$, as

$$\sum_{r=1}^{\infty}\sum_{p_1,\ldots,p_r=0}^{d_r}\sum_{\alpha_1,\ldots,\alpha_r=1}^{s}\boldsymbol{\chi}_{p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_r}(t,x,\boldsymbol{u},\ldots,\boldsymbol{u}_{k-1})D^{p_1}(Q^{\alpha_1})\cdots D^{p_r}(Q^{\alpha_r})$$

where $\boldsymbol{\chi}_{p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_r}(t,x,\boldsymbol{u},\ldots,\boldsymbol{u}_{k-1})$ are analytic on V functions of their arguments, d_r are some natural numbers, and the series are assumed to be convergent on V. This representation of the terms vanishing on $\mathcal{M}$ is more suitable if we want to fix the order n of $\boldsymbol{F}$.

In order to make our results practically applicable we have to find the system of ODEs in t being the result of reduction of the system $\boldsymbol{u}_t=\boldsymbol{F}$ on $\mathcal{M}$. To this end we observe that the general solution of the system made of $\boldsymbol{u}_t=\boldsymbol{F}$ and $\boldsymbol{Q}=0$ should be of the form (4) [17]. Then it is easy to see that $\boldsymbol{u}_t=\boldsymbol{F}$ with $\boldsymbol{F}$ (10) on $\mathcal{M}$ reduces to

$$\sum_{i=1}^{N}\tilde{\boldsymbol{B}}_i(x,c_1(t),\ldots,c_N(t))\frac{\mathrm{d}c_i(t)}{\mathrm{d}t}=\sum_{i=1}^{N}\zeta_i(t,c_1(t),\ldots,c_N(t))\tilde{\boldsymbol{B}}_i(x,c_1(t),\ldots,c_N(t)). \tag{11}$$

The vector function $\boldsymbol{P}(x,t,c_1(t),\ldots,c_N(t))$ in (4) depends in an essential way on all c_i, $i=1,\ldots,N$, by assumption. Hence the vector functions $\tilde{\boldsymbol{B}}_i(x,c_1(t),\ldots,c_N(t))=\partial\boldsymbol{P}(x,c_1(t),\ldots,c_N(t))/\partial c_i$ are linearly independent, and thus (11) reduces to the following system of ODEs for c_i (cf, e.g., [17]):

$$\mathrm{d}c_i(t)/\mathrm{d}t=\zeta_i(t,c_1(t),\ldots,c_N(t))\qquad i=1,\ldots,N. \tag{12}$$

Its general solution depends on N arbitrary constants.

In order to illustrate the above results, consider the following example. Let $s = 1$, so for simplicity we write $\boldsymbol{u} \equiv u$, $\boldsymbol{Q} \equiv Q$, $\boldsymbol{F} \equiv F$. Let $Q = u_2 - a(t)\exp(-u_1)$, where $a(t)$ is a smooth function of t. The general solution of $Q = 0$ is (cf, e.g., p 545 of [30])

$$u = c_2(t) + (x + c_1(t)/a(t))(\ln(a(t)x + c_1(t)) - 1)$$

where $c_1(t)$ and $c_2(t)$ are arbitrary (smooth) functions, and hence we have $h_1 = \exp(u_1) - a(t)x$, $h_2 = u + \exp(u_1)(1 - u_1)/a(t)$. By proposition 1, the most general F such that the equation $u_t = F$ admits $Q\partial/\partial u$ as a generalized conditional symmetry, reads

$$F = x(\dot{a}(t)/a(t))u_1 - (\dot{a}(t)/(a(t))^2)\exp(u_1)(u_1 - 1) + (u_1/a(t))\zeta_1(t, h_1, h_2) + \zeta_2(t, h_1, h_2) + \sum_{p=0}^{m}\sum_{\alpha=1}^{s}\chi_{p,\alpha}(t, x, u, \ldots, u_{j_{p,\alpha}})D^p(u_2 - a(t)\exp(-u_1))$$

where ζ_i and $\chi_{p,\alpha}$ are arbitrary smooth functions of their arguments. If we restrict ourselves to the case $a(t) = \text{const}$ and $F = u_3 + f(x, t, u, u_1, u_2)$, we readily recover one of the results of [31].

2. Linear conditional symmetries and invariant modules

As a further example, illustrating our approach, consider the case when the system $\boldsymbol{Q} = 0$ is a totally nondegenerate linear system of total order $N = \sum_{\alpha=1}^{s} n_\alpha$ of the form

$$u^\alpha_{n_\alpha} = \sum_{\beta=1}^{s}\sum_{j=0}^{n_\beta - 1} g^\alpha_{\beta,j}(x, t)u^\beta_j \qquad \alpha = 1, \ldots, s \tag{13}$$

i.e. $Q^\alpha = u^\alpha_{n_\alpha} - \sum_{\beta=1}^{s}\sum_{j=0}^{n_\beta-1} g^\alpha_{\beta,j}(x, t)u^\beta_j$.

Clearly, the general solution of (13) is $\boldsymbol{u} = \sum_{i=1}^{N} c_i(t)\boldsymbol{f}_i(x, t)$, where $c_i(t)$ are arbitrary functions of t and $\boldsymbol{f}_i \equiv \left(f_i^1, \ldots, f_i^s\right)^T$ are linearly independent solutions of $\boldsymbol{Q} = 0$.

Then we have $h_i = Z_i/Z$, where

$$Z = \begin{vmatrix} f_1^1 & \ldots & f_i^1 & \ldots & f_N^1 \\ \partial f_1^1\big/\partial x & \ldots & \partial f_i^1\big/\partial x & \ldots & \partial f_N^1\big/\partial x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial^{n_1-1} f_1^1\big/\partial x^{n_1-1} & \ldots & \partial^{n_1-1} f_i^1\big/\partial x^{n_1-1} & \ldots & \partial^{n_1-1} f_N^1\big/\partial x^{n_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^s & \ldots & f_i^s & \ldots & f_N^s \\ \partial f_1^s\big/\partial x & \ldots & \partial f_i^s\big/\partial x & \ldots & \partial f_N^s\big/\partial x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial^{n_s-1} f_1^s\big/\partial x^{n_s-1} & \ldots & \partial^{n_s-1} f_i^s\big/\partial x^{n_s-1} & \ldots & \partial^{n_s-1} f_N^s\big/\partial x^{n_s-1} \end{vmatrix}$$

and Z_i are obtained from Z by replacing $\partial^j f_i^\alpha\big/\partial x^j$ by u_j^α (cf, e.g., [28, 29]).

Hence, $\boldsymbol{B}_i = \boldsymbol{f}_i(x, t)$, and $\boldsymbol{R} = \sum_{i=1}^{N}(Z_i/Z)\partial\boldsymbol{f}_i(x, t)/\partial t$, so (10) becomes

$$\boldsymbol{F} = \sum_{i=1}^{N}(Z_i/Z)\partial\boldsymbol{f}_i(x, t)/\partial t + \sum_{i=1}^{N}\zeta_i(t, Z_1/Z, \ldots, Z_N/Z)\boldsymbol{f}_i(x, t) + \sum_{p=0}^{m}\sum_{\alpha=1}^{s}\boldsymbol{\chi}_{p,\alpha}(t, x, \boldsymbol{u}, \ldots, \boldsymbol{u}_{j_{p,\alpha}})D^p(Q^\alpha) \tag{14}$$

where $\chi_{p,\alpha}$ and ζ_i are some smooth functions of their arguments and m and $j_{p,\alpha}$ are some nonnegative integers.

For instance, let $s = 1$, so for simplicity we write $\boldsymbol{u} \equiv u$, $\boldsymbol{Q} \equiv Q$, $\boldsymbol{F} \equiv F$, $\boldsymbol{f}_i \equiv f_i$, and let $Q = u_3 - (a(t))^2 u_1$. Take $f_1 = \exp(a(t)x)$, $f_2 = \exp(-a(t)x)$ and $f_3 = 1$ for the basis of linearly independent solutions of $Q = 0$. $Z = 2(a(t))^3$, $W_2 = \exp(a(t)x)((-a(t))^2 u_1 + a(t)u_2)$, $W_3 = 2a(t)((a(t))^2 u - u_2)$, $h_1 = \exp(-a(t)x)((a(t))u_1 + u_2)/(2(a(t))^2)$, $h_2 = \exp(a(t)x)((-a(t))u_1 + u_2)/(2(a(t))^2)$, $h_3 = u - u_2/((a(t))^2)$, so the most general F such that $u_t = F$ is conditionally invariant under $Q\partial/\partial u$ is of the form

$$F = x(\dot{a}(t)/a(t))u_1 + \zeta_1(t, h_1, h_2, h_3)\exp(a(t)x) + \zeta_2(t, h_1, h_2, h_3)\exp(-a(t)x) \\ + \zeta_3(t, h_1, h_2, h_3) + \sum_{p=0}^{m} \chi_p(t, x, u, \ldots, u_{j_p})(u_{p+3} - (a(t))^2 u_{p+1}).$$

In particular, if $(a(t))^2 = -1$, and we restrict ourselves to the case when $F = F(u, u_1, u_2, u_3)$, then we readily recover one of the results of [24].

More broadly, if we return to the analysis of (14) and assume that $\partial \boldsymbol{f}_i/\partial t = 0, i = 1, \ldots, N, s = 1, \partial \boldsymbol{Q}/\partial t = 0$ and $\partial \boldsymbol{F}/\partial t = 0$ (whence $\partial \chi_{p,\alpha}/\partial t = 0$ and $\partial \zeta_i/\partial t = 0$), then (14) becomes a particular case of the formulae from [29] for $\boldsymbol{F}$ possessing an invariant module spanned by $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_N$. If we further specify to the case $\partial \boldsymbol{F}/\partial x = \partial \boldsymbol{Q}/\partial x = 0$, we also recover some of the results of Fokas and Liu [24].

Note that if we require $\boldsymbol{F}$ (14) to be linear in $\boldsymbol{u}_j$, then (14) simplifies to

$$\boldsymbol{F} = \sum_{i=1}^{N} (Z_i/Z)\partial \boldsymbol{f}_i(x,t)/\partial t + \sum_{i,j=1}^{N} (Z_j/Z)\zeta_{ij}(t)\boldsymbol{f}_i(x,t) + \sum_{p=0}^{m}\sum_{\alpha=1}^{s} \chi_{p,\alpha}(t,x) D^p(Q^\alpha) \qquad (15)$$

where $\chi_{p,\alpha}(t,x)$ and $\zeta_{ij}(t)$ are arbitrary smooth functions.

Using (15), we can, for instance, present a complete classification of Schrödinger-type equations and their multicomponent generalizations in $(1+1)$-dimensional spacetime that reduce to the system of ODEs on the linear space spanned by $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_N$. In particular, we expect that the use of (15) will make it possible to construct new interesting examples of nonstationary quasiexactly solvable Schrödinger-type equations with explicitly time-dependent potentials.

3. Conclusions and discussion

In this paper, we have obtained a complete description of evolution systems (1) admitting a generalized conditional symmetry $\boldsymbol{Q}(x, t, \boldsymbol{u}, \ldots, \boldsymbol{u}_k)\partial/\partial \boldsymbol{u}$ under the assumption that the system of ODEs $\boldsymbol{Q} = 0$ is analytic and totally nondegenerate. This generalizes the earlier work of Doyle [27], Svirshchevskii [21–23] and, partially, of Fokas and Liu [24] and of Kamran, Milson and Olver [29], to the case when $\partial \boldsymbol{Q}/\partial t \neq 0$. Our results can be further applied for the construction of exactly solvable initial value problems in the spirit of Basarab-Horwath and Zhdanov [31, 32] and for the classification of generalized symmetries (and then of the evolution systems admitting these symmetries) compatible with the prescribed boundary conditions, using the ideas of Adler *et al* [33]. Another interesting possibility is the construction of explicitly time-dependent quasiexactly solvable models, cf the discussion in previous section and in [29]. Finally, it would be of great interest to generalize the results of this paper to the case of multiple space dimensions. We intend to analyse these topics in more detail elsewhere.

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